Simultaneous Linear Equations

Setting Up Problem in Matrix Form

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Introduction

- Matrix algebra is used to solve a system of simultaneous linear equations.
- In fact, for many mathematical procedures such as the solution to a set of nonlinear equations, interpolation, integration, and differential equations, the solutions reduce to a set of simultaneous linear equations.
- Let us illustrate with an example.

Example

- Set up the following system of equations in a matrix format

$$\begin{aligned} 25 + 5y + z &= 106.8 \\ 64x + 8y + z &= 177.2 \\ 144x + 12y + z &= 279.2 \end{aligned}$$

- This set of equations can be rewritten in the matrix form as

$$\begin{bmatrix} 25x + & 5y + & z \\ 64x + & 8y + & z \\ 144x + & 12y + & z \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example

- The above equation can be written as a linear combination as follows:

$$x\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + y\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

and further using matrix multiplication gives

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Matrix representation of simultaneous equation

- A general set of m linear equations and n unknowns,

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = c_2 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m
\end{array}
$$

can be rewritten in the matrix form as

$$
\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}
$$

Matrix form

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

- Denoting the matrices by [A], [X], and [C], the system of equation is [A] [X]=[C], where
  - [A] is called the coefficient matrix,
  - [C] is called the right-hand side vector, and
  - [X] is called the solution vector

Augmented form

- Sometimes [A] [X]=[C] system of equations is written in the augmented form, that is,

$$[A \,\vdots\, C] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & c_2 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & c_n \end{bmatrix}$$

- Example:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 25 & 5 & 1 & 106.8 \\ 64 & 8 & 1 & 177.2 \\ 144 & 12 & 1 & 279.2 \end{array}\right]$$

How to solve for our unknowns?

- [A] [X]=[C]

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

- We use matrix inverse; next video

Conclusion

- Matrix representation of linear system of equations

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = c_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = c_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m
\end{aligned}
\Rightarrow
\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}
$$