Linear Circuits and Systems

Complex Numbers– Part 2 of 2

Course: Linear Circuits and Systems
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Objectives

This series will focus on:

- Review of Complex Numbers
  - Part 1:
    - The *Real* and *Imaginary* parts
    - Rectangular Form
    - Polar Form
    - Graphic Form
  - Part 2:
    - → Arithmetic Operations
    - Examples

Addition and Subtraction $z = x + jy$

- Adding and Subtracting complex numbers is easiest if the numbers are in rectangular form.

Steps:

1. Write each complex number in the form $a + jb$.
2. Add or subtract the real parts of the complex numbers.
3. Add or subtract the imaginary parts of the complex numbers.

- Add

$$(a + jb) + (c + jd) = (a + c) + (jb + jd) = (a + c) + j(b + d)$$

- Subtract

$$(a + jb) - (c + jd) = (a + jb) + (-c - jd) = (a - c) + (jb - jd)$$
$$= (a - c) + j(b - d)$$

Addition Examples:

Add $(11 + j5) + (8 - j2)$ $= (11+8) + j(5-2) = 19 + j3$

$= (11 + 8) + (j5 - j2)$

$\boxed{= 19 + j3}$

$j = \sqrt{-1}$

Add $(10 + \sqrt{-5}) + (21 - \sqrt{-5})$ $= (10 + j\sqrt{5}) + (21 - j\sqrt{5})$

$= (10 + j\sqrt{5}) + (21 - j\sqrt{5})$ $= (10+21) + j(\cancel{\sqrt{5}} - \cancel{\sqrt{5}})$

$= (10 + 21) + (j\sqrt{5} - j\sqrt{5})$ $= 31$

$\boxed{= 31}$

Multiplication in Polar Form

(exponential)

- Multiplying complex numbers is easiest if the numbers are in polar form.
- Suppose $z_1 = r_1 \angle \phi_1$ and $z_2 = r_2 \angle \phi_2$ $z_1 = r_1 e^{j\phi_1}$ $z_2 = r_2 e^{j\phi_2}$

  Then $z_1 \times z_2 = (r_1 \times r_2) \angle (\phi_1 + \phi_2)$ $z_1 \times z_2 = r_1 r_2 e^{j(\phi_1 + \phi_2)}$

- In words: to multiply two complex numbers in polar form, multiply their magnitudes to get the magnitude of the result, and add their angles to get the angle of the result.

Multiplication in Rectangular Form

$j = \sqrt{-1}$

$j^2 = -1$

$ac + jad + jbc + j^2 bd$

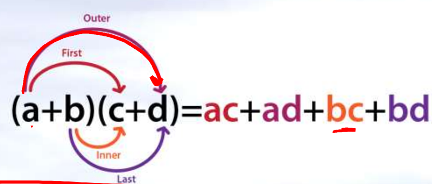
$$(a + jb)(c + jd) = ac + jad + jbc + j^2bd$$

$-bd$

$$= ac + jad + jbc + (-1)bd$$

$$= (ac - bd) + j(ad + bc)$$

1. Use the FOIL method to find the product.



2. $\boxed{\text{Replace } j^2 \text{ by } -1}$.

3. Write the answer in the form $a + jb$.

Ex: Multiply Complex Numbers

FOIL

$i = \sqrt{-1}$ (j)

$i^2 = -1$

$$(5 - 2i)(4 + i) = 5*4 + 5i - 8i - 2i^2$$

$$20 + 5i - 8i - 2i^2$$

$$20 - 3i - 2i^2$$

$$20 - 3i - 2(-1)$$

(-1)

$$20 - 3i + 2$$

$$22 - 3i$$

Complex Conjugate

z z^*

- Given a complex number in rectangular form,

  $\rightarrow$ $z = x + jy$

  its **complex conjugate** is simply

  $z^* = x - jy$

- Given a complex number in polar form,

  $z = r\angle\phi$ $z = re^{j\phi}$

  its **complex conjugate** is simply $z^* = re^{-j\phi}$

  $z^* = r\angle -\phi$

Complex Conjugate

The product of conjugates is the real number $a^2 + b^2$.

$(a + jb)(a - jb) = a^2 - j^2b^2$ $(r\angle\phi)(r\angle-\phi) = r^2$

$r^2\angle\phi-\phi = r^2\angle 0 = r^2$

$= a^2 - (-1)b^2$

$= a^2 + b^2 \equiv r^2$ $\boxed{r = \sqrt{a^2+b^2}}$

Example: $(5 + j2)(5 - j2) = (5^2 - j^2 4)$

$5^2 + 2^2$

$25 + 4 = 29$

$= 25 - (-1)\,4$

$= 29$

Example: $(5\angle 45)(5\angle -45) = 25$

Division in Polar Form

- Dividing complex numbers is also easiest if the numbers are in polar form.
- Suppose $z_1 = r_1\angle\phi_1$ and $z_2 = r_2\angle\phi_2$ $Z_1 = r_1 e^{j\phi_1}$ $Z_2 = r_2 e^{j\phi_2}$

  Then $z_1 \div z_2 = (r_1 \div r_2)\angle(\phi_1 - \phi_2)$ $\frac{Z_1}{Z_2} = \frac{r_1 e^{j\phi_1}}{r_2 e^{j\phi_2}} = \frac{r_1}{r_2} \underbrace{e^{j(\phi_1-\phi_2)}}_{e^{j(\phi_1-\phi_2)}}$

- In words: to divide two complex numbers in polar form, divide their magnitudes to get the magnitude of the result, and subtract their angles to get the angle of the result.

Division in Rectangular Form

$\frac{a+jb}{c+jd} * \frac{c-jd}{c-jd}$

A rational expression, containing one or more complex numbers, is in simplest form when there are no imaginary numbers remaining in the denominator.

Example:

$$\frac{5+3i}{2+i} * \frac{(2-i)}{(2-i)}$$

$$= \frac{5+3i}{2+i}\cdot\frac{2-i}{2-i} = \frac{10-5i+6i-3i^2}{2^2+1^2}$$

$$= \frac{10+i-3(-1)}{4+1} = \frac{13+i}{5} = \frac{13}{5} + \frac{1}{5}i$$

Example:

$z_1 = 6\angle 26.57$

$z_2 = 3\angle -18.43$

$$\frac{z_1}{z_2}$$

$$\frac{(6\angle 26)}{(3\angle -18)} = 2\angle 45$$

Conclusion

Acknowledgement

This material is based upon work partially supported by the National Science Foundation under Grant# 2335802. Any opinions, findings, and conclusions, or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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