Linear Circuits and Systems

Matrix Inverse

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Reminder: Matrix form

- Matrix representation of linear system of equations

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_m \end{bmatrix}$$

A $\quad [A]X = C$

- How to solve for X?

[X]=[C]

Can we divide two matrices?

- If $[A]\,[\mathrm{X}] = [C]$ is defined, it might seem ~~intuitive~~ that $[X] = \frac{[C]}{[A]}$, but matrix division is not defined like that.

- The inverse of a square matrix $[A]$, if existing, is denoted by $[A]^{-1}$ such that $[A]\,[A]^{-1} = [I] = [A]^{-1}[A]$
  - where $[I]$ is the identity matrix.

- As a result, we can solve for $[\mathrm{X}]$ as follows
$$\boxed{[\mathrm{X}] = [A]^{-1}[C]}$$

Solution to a set of equations, [A][X] = [C]

- If the number of equations is the same as the number of unknowns, the coefficient matrix [A] is a square matrix.

X

Given

$$A^{-1} \rightarrow \quad [A]\,[X] = [C]$$

Then, if $[A]^{-1}$ exists, multiplying both sides by $[A]^{-1}$.

$$\rightarrow \boxed{[A]^{-1}[A]}[X] = [A]^{-1}[C]$$

$$\rightarrow [I]\,[X] = [A]^{-1}[C]$$

$$\boxed{[X] = [A]^{-1}[C]}$$

This implies that if we are able to find $[A]^{-1}$, the solution vector of $[A]\,[X] = [C]$ is simply a multiplication of $[A]^{-1}$ and the right-hand side vector, $[C]$.

Matrix inverse

- Let A be a square matrix. If $[B]$ is another square matrix of the same size such that $[B]\,[A] = [I]$, then $[B]$ is the inverse of $[A]$.
  - $[A]$ is then called to be invertible or nonsingular.
- If $[A]^{-1}$ does not exist, $[A]$ is called noninvertible or singular.

Matrix inverse

If $[A]$ and $[B]$ are two $n \times n$ matrices such that $[B]\,[A] = [I]$, then these statements are also true

a) $[B]$ is the inverse of $[A]$
b) $[A]$ is the inverse of $[B]$
c) $[A]$ and $[B]$ are both invertible
d) $[A][B] = [I]$.
e) $[A]$ and $[B]$ are both nonsingular
f) all columns of $[A]$ and $[B]$ are linearly independent
g) all rows of $[A]$ and $[B]$ are linearly independent.

Example

- Determine if

$$[B] = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

is the inverse of

$$[A] = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

$B * A = A^{-1} \times A = I$

if $B = A^{-1}$

$$\begin{bmatrix} 3 & 2 \\ 5 & \boxed{3} \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & \boxed{-3} \end{bmatrix} = \begin{bmatrix} -9+10 & 6-6 \\ -15+15 & 10-9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Inverse of a 2x2 matrix

- Given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Example

$$[A] = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\underbrace{9 - 10}_{-1}}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Wrap it up: Solution of Simultaneous Equations using Matrix Method

$$AX = C$$

$$X = A^{-1}C$$

Example

- Solve the following system using the matrix method (1)
  - 2x+y = 4 (2)
  - x-y = -1

step 1: put eqⁿ into Matrix format

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad ; \underline{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A \quad X = C$$

$$\boxed{X = A^{-1}C}$$

step 2: find A^{-1}

$$A^{-1} = \frac{1}{-2-1} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

step 3: $X = A^{-1}C$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \leftarrow \quad \underline{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{x = 1, \; y = 2}$$

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